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ASYMMETRIC STRETCHING OF A SYMMETRICALLY LOADED ELASTIC SHEET

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Abstract—This consideration of the stretching of a square of thin elastic sheeting by equal tensile forces acting in dead loading on the edges of the square is motivated by some early observations of Treloar. For any reasonable elastic materials the loading will certainly correspond to some symmetric stretching but, it is shown here that depending on the form of the strain energy function and for some values of load an asymmetric stretching can equilibrate the symmetric loading as well as the more obvious symmetric stretching. Furthermore, the asymmetric response, when it is a dynamical possibility, is stable while the symmetric response is unstable. The phenomenon does not occur for neo-Hookean rubbers but does occur for the elastic materials corresponding to many realistic strain energies, Mooney-Rivlin rubbers for example.

1. INTRODUCTION

Because of the severe nonlinearities inherent in finite elasticity, the application of dead loading to an elastic body often leads to multiple possibilities for the resulting deformations. It is not uncommon in such cases for some of the possible deformations to show an unexpected lack of symmetry. A well-known early theoretical example was given by Rivlin[1], who examined homogeneous deformations of a unit cube of incompressible neo-Hookean elastic material subjected to equal tensile forces, normal to each of its faces. Rivlin showed that at least seven equilibrium configurations are possible when the tensile forces exceed a certain critical value, and of the seven, only one, the undeformed configuration, has all the symmetries of the given loading. Furthermore, on examination of the stabilities of the equilibria[2] for large values of the tensile forces, he found that the symmetric configuration is unstable. Most people on first encountering this result find it somewhat surprising.

Much less well known is an experimental example of this sort of asymmetric solution to a dead-loading problem, which was observed at about the same time as Rivlin's theoretical example. In the course of experiments on rubber, Treloar[3] stretched a square of thin rubber sheeting by equal tensile forces acting on the edges of the square, and observed that in some cases of high loading, the principal stretches corresponding to the equal tensile loads were not equal. For experimental reasons the stretches in these experiments were limited to values less than three. But despite this limitation, anisotropies approaching fifteen percent were observed in the extreme cases; that is, the two stretches differed by that amount. Treloar reported this odd behavior through data without comment, because he was primarily interested in the failure of the data to conform to certain predictions for neo-Hookean elastic sheets. Indeed, at the time, the failure of the neo-Hookean model of rubber elasticity was unexpected enough that Treloar[4] was careful to consider various experimental artifacts which might have led to his results, and conclusively eliminate the possibility in each case. For the most part, that discussion also confirms the occurence of the phenomenon we consider here. The phenomenon was hardly noticed at the time, however, and since then seems to have been largely forgotten or ignored; probably because shortly after Treloar's revelation, convenient force-measuring devices of high mechanical impedance became available. The techniques for measuring biaxial stretching of rubber sheets changed accordingly and currently dead loading is seldom used for that purpose.

Shield[5] has examined the stability of uniformly stretched elastic membranes under dead loading by calculating to second order the effect on strain energy of a small displacement superposed on an initial finite deformation. His analysis reveals instability in certain cases of biaxial stretching of an elastic sheet with a suitable strain-energy function, which he might have used to explain Treloar's observations of asymmetry for rubber under symmetric loading. However, Shield was probably unaware of these data and chose a strain energy of an empirical form which did not generate the asymmetry. He therefore did not examine this possibility in greater detail.

In this work we shall analyze the phenomenon observed by Treloar. A simple elasticity problem, modeling Treloar's experiment, will be shown to lead to multiple solutions with asymmetry. For a Mooney–Rivlin type of material, or for other more realistic materials, this asymmetric solution must be expected to occur when sufficiently large stresses can be supported by the elastic sheet. On the other hand, for neo-Hookean rubber, the model does not lead to asymmetric solutions, and thus the mechanism is distinct from that displayed by Rivlin's cube.

2. THE BASIC EQUILIBRIA

Consider a unit square, cut from a thin elastic sheet, stretched and held in equilibrium by tensile forces applied uniformly in the plane of the sheet and normal to the cut edges. The square surfaces of the sheet are traction free. We consider homogeneous isochoric deformations which take the square into a rectangle by pure stretching. If the sheet is composed of incompressible isotropic material with a strain-energy function, W, we can write down the following equation for the stress acting in the direction λ,

$$\sigma_\lambda = \lambda^2 W_1 - \frac{1}{\lambda^2} W_2 + p, \tag{2.1}$$

where σ_λ is the stress, W_1 and W_2 are the derivatives of the strain-energy function with respect to the first and second invariants of the strain, and p is an arbitrary isotropic stress associated with the constraint of incompressibility. Similar equations apply for the stress in the direction of the other two principal stretches. In these equations, W and its derivatives are defined by the following relations:

$$W = f(\mathrm{I}, \mathrm{II}), \qquad W_1 = \frac{\partial f}{\partial \mathrm{I}}, \qquad W_2 = \frac{\partial f}{\partial \mathrm{II}}, \tag{2.2}$$

$$\mathrm{I} = \lambda^2 + \mu^2 + \frac{1}{\lambda^2\mu^2}, \qquad \mathrm{II} = \frac{1}{\lambda^2} + \frac{1}{\mu^2} + \lambda^2\mu^2, \tag{2.3}$$

where λ and μ are principal stretches in the plane of the sheet. Because the material is incompressible the stretch in the thickness direction is $1/(\lambda\mu)$.

No tractions act on the square surfaces of the sheet, and therefore the force in the thickness direction must be zero. This fact allows us to evaluate the quantity p as follows:

$$p = \lambda^2\mu^2 W_2 - \frac{1}{\lambda^2\mu^2} W_1. \tag{2.4}$$

The force acting in the λ direction, f_λ, is given by

$$f_\lambda = \frac{t}{\lambda}\sigma_\lambda, \tag{2.5}$$

where t is the thickness of the original undeformed unit-square sheet. A similar equation gives the force in the μ direction.

From eqns (2.1), (2.4) and (2.5), and the similar equations for the μ direction, we can write the following expressions for the forces acting on the edges of the unit square:

$$f_\lambda = t\,\frac{\lambda^4\mu^2 - 1}{\lambda^3\mu^2}\,(W_1 + \mu^2 W_2), \tag{2.6a}$$

$$f_\mu = t\,\frac{\lambda^2\mu^4 - 1}{\lambda^2\mu^3}\,(W_1 + \lambda^2 W_2). \tag{2.6b}$$

We now stipulate that the loads on the edges of the sheet are equal in magnitude, so that we can equate f_λ with f_μ and factor out the thickness from the resulting equation to get

$$(\lambda - \mu)[(\lambda^3\mu^3 + 1)W_1 + (\lambda^2 + \lambda\mu + \mu^2 - \lambda^4\mu^4)W_2] = 0. \tag{2.7}$$

One root of this equation is the symmetric solution, $\lambda = \mu$. This solution describes a simple equibiaxial stretch, taking the unit square into another square.

Of greater interest are deformations corresponding to the vanishing of the second factor in eqn (2.7); that is, the roots of the following equation:

$$(\lambda^3\mu^3 + 1)W_1 + (\lambda^2 + \lambda\mu + \mu^2 - \lambda^4\mu^4)W_2 = 0. \tag{2.8}$$

When W_2 is not zero we can express eqn (2.8) in the following form:

$$\frac{W_1}{W_2} \equiv K = \frac{\lambda^4\mu^4 - (\lambda^2 + \lambda\mu + \mu^2)}{\lambda^3\mu^3 + 1}, \tag{2.9}$$

which should be compared with inequality (4.8) of Shield[5].

It is obvious that the right side of this equation is always less than $\lambda\mu$, and thus if K is equal to or greater than $\lambda\mu$, there are no roots possible. Certainly for a neo-Hookean material (for which W_2 is zero) no roots are possible. Therefore a neo-Hookean material can only respond to this symmetric loading (within the limitation of homogeneous deformations) by deforming symmetrically. If the ratio K is always positive and finite, however, this equation will have roots determining a curve in the λ, μ plane; each point on the curve representing a homogeneous deformation which equilibrates equal tensile loads on the edges of the unit square.

We shall now consider the consequences when K, the ratio of the derivatives of the strain-energy function, is a constant; a restriction which includes materials of the Mooney–Rivlin type. Indeed, for K to be a constant it is necessary and sufficient that the strain-energy be solely a function of the variable I + II K. The Mooney–Rivlin strain energy corresponds to the simple case of proportionality. When K is a positive finite constant, eqn (2.9) defines a curve in the plane of λ, μ. The hyperboloid of Fig. 1 labeled $K = 5$ is an example. It is more interesting, however, to look at the corresponding curve in the plane of the deformation invariants. In that plane each possible homogeneous deformation is represented by a unique point within a region of the plane bounded by two smooth arcs forming a cusp[6] at the point (3,3). This construction can be seen in Fig. 2. The upper arc represents all states of equal biaxial stretching of the sheet, and it is thus the locus of symmetric roots of eqn (2.7) for tensile loads. For isotropic incompressible elastic materials, this curve of symmetric solutions does not depend upon details of the strain-energy function. Each point along this curve corresponds to a stretching force determined by eqn (2.6a) or, equivalently in this case, eqn (2.6b) and the condition $\lambda = \mu$, thus

$$\frac{f}{tW_2} = \frac{\lambda^6 - 1}{\lambda^5}\,(K + \lambda^2). \tag{2.10}$$

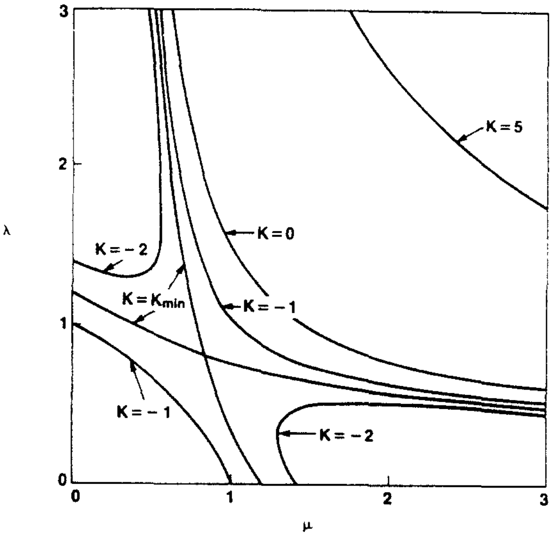

Fig. 1. Loci of asymmetric equilibria on the λ-μ plane for various values of K.

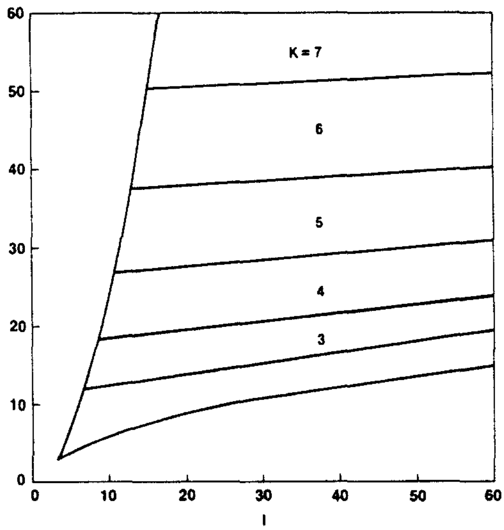

Fig. 2. Asymmetric solutions in the I-II plane for K = 3, 4, 5, 6, 7.

For a given λ the point along the curve in the plane of invariants can be found from eqns (2.3) and the condition $\lambda = \mu$; that is,

$$I = 2\lambda^2 + \frac{1}{\lambda^4}, \qquad II = \lambda^4 + \frac{2}{\lambda^2}. \tag{2.11}$$

For any positive finite value of constant K another curve in the plane of the invariants, the locus of asymmetric equilibria, is determined by eqn (2.9). This curve will generally intersect the curve of symmetric equilibria at a point which depends on the value of K. By setting $\lambda = \mu$ in eqn (2.9), we can find a relation for the stretch at this intersection:

$$K = \frac{\lambda^2(\lambda^6 - 3)}{\lambda^6 + 1}. \tag{2.12}$$

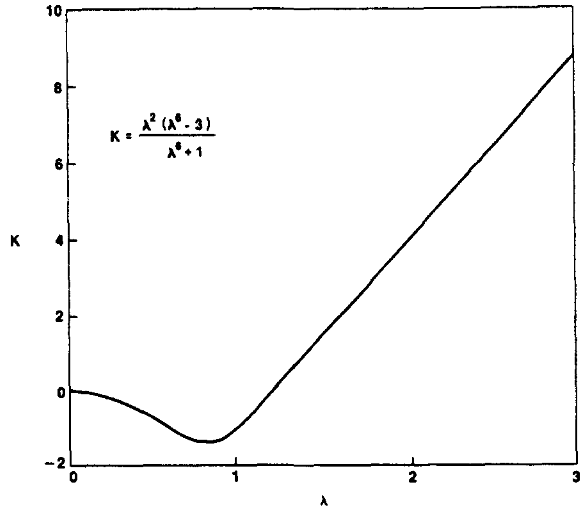

Fig. 3. Plot for finding λ at intersection of loci of symmetric and asymmetric equilibria.

Figure 3 is a plot of the right-hand side of this equation as a function of stretch. Even for a strain energy with a nonconstant K, if we were to superimpose on this figure a plot of K as a function of equal biaxial stretch, the intersections of the two plots would mark values of stretch for which we should expect curves of asymmetric solutions to branch off. That is, at the corresponding point in Fig. 2, we should expect a curve of equilibria to branch from the boundary into the interior of the cuspoidal region. Such curves are displayed in Fig. 2 for a range of positive constant values of K.

It is interesting to repeat this analysis for a strain-energy function expressed in terms of the principal stretches rather than the strain invariants. The analog of eqn (2.8) is in this case

$$\frac{W_\lambda - W_\mu}{\lambda - \mu} + \frac{1}{\lambda^2\mu^2} W_\nu = 0 \tag{2.13}$$

where

$$W_\lambda = \frac{\partial W(\lambda, \mu, \nu)}{\partial \lambda},$$

etc. When the strain-energy function is of the Valanis–Landel form[7] for which

$$W = w(\lambda) + w(\mu) + w\left(\frac{1}{\lambda\mu}\right), \tag{2.14}$$

where $w(\ )$ is a constitutive function called the V–L function[8], eqn (2.8) becomes

$$\frac{w'(\lambda) - w'(\mu)}{\lambda - \mu} + \frac{1}{\lambda^2\mu^2} w'\left(\frac{1}{\lambda\mu}\right) = 0, \tag{2.15}$$

where a prime indicates a derivative of the V–L function. Pairs of stretches, (λ, μ), which are roots of this equation determine asymmetric solutions of the symmetric-loading problem. The branches of asymmetric solutions intersect the curve of symmetric solutions at a point given by this equation in the limit as μ approaches λ: an equation analagous eqn (2.12), viz.

$$w''(\lambda) + \frac{1}{\lambda^4} w'\left(\frac{1}{\lambda^2}\right) = 0, \tag{2.16}$$

where the double prime indicates a second derivative. Of course, since a Mooney–Rivlin strain-energy function may be expressed in a Valanis–Landel form by setting

$$w(\lambda) = C_1(\lambda^2 - 2\log\lambda - 1) + C_2\left(\frac{1}{\lambda^2} + 2\log\lambda - 1\right) \tag{2.17}$$

if this form is used in eqn (2.15) and K is substituted for C_1/C_2, one obtains eqn (2.9). The constant terms and the logarithmic terms are included in the V–L function as a convention: they do not lead to any physical consequences[7].

3. EQUILIBRIA WHEN K IS NEGATIVE

It is evident from Fig. 3 that for some values of K there are two intersections of branches of asymmetric solutions with the curve of symmetric solutions. This occurs for values of K between zero and the minimum value on the curve of Fig. 3. The minimum can be calculated from the derivative of eqn (2.12), which gives

$$\lambda_{\min} \simeq 0.81428, \qquad K_{\min} \simeq -1.3905. \tag{3.1}$$

Within this range of values of K, eqn (2.9) determines a curve of two branches in the plane of λ and μ. Figure 1 shows these two branches for the case $K = -1$, curves which are typical for this range of K. Notice that one of these branches intersects the curve of symmetric deformations at a stretch of less than one. That is to say, the asymmetric solutions occur for compressive forces, as well as tensile forces.

When K is at its minimum value in Fig. 3, there is again only one branch point from the curve of symmetric solutions. In this case two branches of asymmetric solutions osculate at the intersection with the curve of symmetric solutions. Figure 1 illustrates this also.

For values of K below the minimum value of Fig. 3, there are no physically interesting solutions of eqn (2.12). The asymmetric solutions fall on two branches which do not intersect the curve of symmetric solutions. Figure 1 illustrates these branches in the case $K = -2$.

It does not seem to make physical sense for K to be negative for all values of the stretches. Consider, for instance, that according to eqns (2.6), when the square of a stretch is equal to $-K$, the corresponding force is zero. Indeed the situation for negative K appears to be of less practical interest than that for positive K. In general, typical rubbers in the deformations commonly observed seem to display nonconstant values of K which are positive and range between limited values. In a typical example[8], this range might be from 4 to 10. Some observations of rubber at very small deformations have suggested that K may possibly take on negative values in extreme cases[9]. Other observations on torsion of plastic cylinders suggest that these data also lead to negative values of K when analyzed in terms of time-dependent elasticity[10]. Therefore the subject is not completely devoid of interest. We will not pursue it further, however, in this work.

4. STABILITY

Considerations of stability tell us which of the possible equilibria will actually occur when forces are applied. The symmetries of the problem in our case offer a particularly simple way to see that for constant positive finite K when, in terms of invariants, only one asymmetric equilibrium is possible, that the asymmetric equilibrium is stable and that the symmetric equilibrium is unstable. Let us assume, as seems reasonable, that

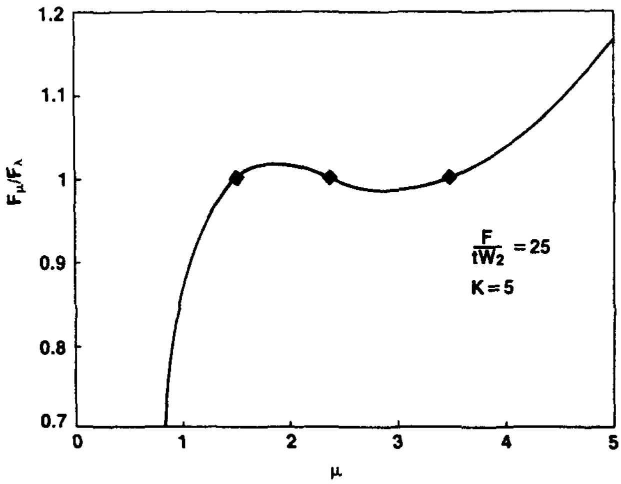

Fig. 4. Force ratio, F_μ/F_λ, for F_λ fixed and μ varied.

the stretch of the symmetric deformation in equilibrium with two equal forces, f_0, is bracketed by the two stretches of the asymmetric equilibrium under the same symmetric loading; that is, if λ_s is the stretch in the symmetric case and λ_a, μ_a are the stretches in the asymmetric case, then

$$\lambda_a > \lambda_s > \mu_a. \tag{4.1}$$

We imagine the square elastic sheet loaded in one direction with a fixed load f_0, while the other direction is constrained to sweep through a range of stretches, including the range between the values λ_a and μ_a. If the ratio of the force needed to produce these stretches to the fixed force f_0 is plotted against the stretch, we obtain a curve such as that shown in Fig. 4. The curve can intersect the line $f/f_0 = 1$ at only three points, since by hypothesis there are only three different stretches which occur in equilibria under this loading. Realistically, at μ_a, the curve must be rising from the state of simple extension and at λ_a the curve must also be rising to an infinite force for infinite extension. Excluding the highly unrealistic circumstance that the curve has slope zero at these intersections, it must cross at λ_s with a negative slope. Figure 4 indicates that the application of a force f_0 to the constrained side will cause it to move from the symmetric solution at λ_s to the asymmetric solution at λ_a. On the other hand, a force of f_0 is not sufficient to stretch the sheet from the asymmetric solution at μ_a to the symmetric solution at λ_s.

This discussion of stability suggests that for any realistic strain-energy function, the asymmetric solution will occur whenever it is a possible equilibrium. Of course, the argument is perhaps a trifle naive; for instance we have not examined the possibility of inhomogeneous deformations or equilibria for which the principal stretches are not in the direction of the forces. Chen[11] has outlined a deft method for improving this argument based on some of his results. Alternatively, Shield's analysis[5] shows also that in these circumstances the symmetric equilibria are unstable. Furthermore, if the strain energy is known, we can actually calculate the curve of Fig. 4 and demonstrate the instability directly. Figure 4 is, in fact, a plot of such a calculation for a Mooney–Rivlin material with $K = 5$ and for a force such that $F = 25$ and λ_s is 2.3662. With the force f_0 in the λ direction, we calculate μ as a function of λ to get

$$\mu^2 = \frac{\sqrt{(A^2 + 4K\lambda^4)} + A}{2\lambda^4}, \tag{4.2}$$

where

$$A = \frac{f_0\lambda^3}{2C_2 t} - K\lambda^4 + 1, \tag{4.3}$$

and C_2 is the coefficient of II in the Mooney–Rivlin strain-energy function. From eqn (2.6) we can calculate the ratio f/f_0 to get

$$\frac{f}{f_0} = \frac{\lambda}{\mu}\frac{\lambda^2\mu^4 - 1}{\lambda^4\mu^2 - 1}\frac{K + \lambda^2}{K + \mu^2}. \tag{4.4}$$

Equations (4.2), (4.3) and (4.4) can be used to plot this ratio against μ as we have done in Fig. 4.

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5. A CONVENIENT CALCULATION OF ASYMMETRIC EQUILIBRIA

From eqn (2.6) we have

$$F_\lambda \equiv \frac{f_\lambda}{tW_2} = \frac{\lambda^4\mu^2 - 1}{\lambda^3\mu^2}(K + \mu^2), \tag{1aA}$$

$$F_\mu \equiv \frac{f_\mu}{tW_2} = \frac{\lambda^2\mu^4 - 1}{\lambda^2\mu^3}(K + \lambda^2). \tag{1bA}$$

When $f_\lambda = f_\mu$, then $F_\lambda = F_\mu = F$ and we can eliminate K between eqns (1A), to get

$$F = \frac{(\lambda + \mu)(\lambda^4\mu^2 - 1)(\lambda^2\mu^4 - 1)}{\lambda^2\mu^2(\lambda^3\mu^3 + 1)}. \tag{2A}$$

On the other hand, if we eliminate F from eqns (1A), we get

$$\lambda^2 + \mu^2 = \lambda\mu(\lambda^3\mu^3 - 1) - K(\lambda^3\mu^3 + 1) \tag{3A}$$

or, equivalently

$$(\lambda + \mu)^2 = (\lambda\mu - K)(\lambda^3\mu^3 + 1). \tag{4A}$$

In each of eqns (2A), (3A) and (4A) the quantity $\lambda - \mu$, and hence the symmetric equilibrium, has been factored out. By combining eqns (2A) and (3A) we can generate the concise equation

$$F = (\lambda + \mu)\frac{1 + K\lambda^2\mu^2}{\lambda^2\mu^2}, \tag{5A}$$

which is valid for asymmetric equilibria. We can generate an equation expressing F as a function of K and $\lambda\mu$ alone by combining eqns (4A) and (5A), viz.

$$F = \sqrt{[(\lambda\mu - K)(\lambda^3\mu^3 + 1)]}\,\frac{1 + K\lambda^2\mu^2}{\lambda^2\mu^2}. \tag{6A}$$

With this equation, a value of $\lambda\mu$ can be found corresponding to given values of F and K. Let this value of $\lambda\mu$ be designated as b, then we have the following:

$$\lambda\mu = b, \qquad \lambda + \mu = \sqrt{[(b - K)(b^3 + 1)]}. \tag{7A}$$

If we eliminate μ between the eqns (7A), we obtain a quadratic equation in λ, namely

$$\lambda^2 - \sqrt{[(b - K)(b^3 + 1)]}\lambda + b = 0. \tag{8A}$$

This equation can easily be solved to give

$$\lambda = \frac{\sqrt{[(b - K)(b^3 + 1)]} + \sqrt{[b(b^3 - 3) - K(b^3 + 1)]}}{2}, \tag{9aA}$$

$$\mu = \frac{\sqrt{[(b - K)(b^3 + 1)]} - \sqrt{[b(b^3 - 3) - K(b^3 + 1)]}}{2}. \tag{9bA}$$

Equations (9A) may also be viewed as parametric equations for plotting the asymmetric solutions for constant K as in Fig. 1. Analogous equations for plotting in the plane of deformation invariants I and II are

$$\mathrm{I} = b(b^3 - 1) - K(b^3 + 1) + \frac{1}{b^2}, \tag{10aA}$$

$$\mathrm{II} = \frac{b(b^3 - 1) - K(b^3 + 1)}{b^2} + b^2. \tag{10bA}$$